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Hamiltonian for a Relativistic Particle with Linear Dissipation

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A general formalism for obtaining the Lagrangian and Hamiltonian for a one dimensional dissipative system is developed. The formalism is illustrated by applying it to the case of a relativistic particle with linear dissipation. The hamiltonian for a relativistic particle with linear dissipation is obtained. An example of this approach is given.

KEY WORDS: Lagrangian; Hamiltonian; dissipative system.
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1. INTRODUCTION

Dissipative systems has been one of the must subtle and difficult topics to deal with in classical and quantum physics. It is very well known that for non dissipative systems the Lagrangian and Hamiltonian can be easily obtained by subtracting or adding respectively the kinetic and potential energy of the system (Goldstein, 1980), but when we have dissipation in our dynamical system this construction is not useful and the corresponding Lagrangian and Hamiltonian is certainly not trivial to obtain (González, 2004; López, 1996). The reason is that there is not yet a consistent Lagrangian and Hamiltonian formulation for dissipative systems. The problem of obtaining the Lagrangian and Hamiltonian from the equations of motion of a mechanical system is a particular case of "The Inverse problem of the Calculus of Variations" (Santilli, 1978; Vujanovic and Jones, 1989). This topic has been studied by many mathematicians and theoretical physicists since the end of the last century. The interest of physicists in this problem has grown recently because of the quantization of dissipative systems. A mechanical system can be quantized once its Hamiltonian is known and this Hamiltonian is usually obtained from a Lagrangian. The problem of quantizing dissipative systems has

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been extensively studied for non-relativistic systems (Um *et al.*, 2002; López and González, 2004) but little has been done for relativistic systems.

The main purpose of this paper is to develop a general formalism to obtain Lagrangians and Hamiltonians for one dimensional dissipative systems and apply it to the case of a relativistic particle under the action of a dissipative force which is proportional to its velocity.

2. LAGRANGIAN AND HAMILTONIAN FOR DISSIPATIVE SYSTEMS

Newton's equation of motion for one dimensional systems can be written as the following dynamical system

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = F(t, x, v), \tag{1}$$

where t is time, x is the position of the particle, v is the velocity and $F(t, x, v)$ is the force divided by the mass of the particle. If a Lagrangian function $L(t, x, v)$ is given for (1) then the Hamiltonian of the system can be obtained by the Legendre transform

$$H(t, x, p) = pv(t, x, p) - L(t, x, v(t, x, p)), \tag{2}$$

where $v(t, x, p)$ is the inverse function of the generalized linear momentum given by

$$p = \frac{\partial L}{\partial v}(t, x, v),$$

which enables us to work with the so called canonical variables, unfortunately since it is not always possible to obtain explicitly $v(t, x, p)$ then not every dynamical system has a Hamiltonian function (González, 2004).

It is well known that the Lagrangian L for a one dimensional system always exist and must satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial x}. \tag{3}$$

Expanding (3) and differentiating with respect to v we obtain

$$v\frac{\partial^3 L}{\partial v^2 \partial x} + F\frac{\partial^3 L}{\partial v^3} + \frac{\partial^3 L}{\partial v^2 \partial t} + \frac{\partial F}{\partial v}\frac{\partial^2 L}{\partial v^2} = 0. \tag{4}$$

Therefore, in order to obtain the Lagrangian of (1) we have to find a nontrivial solution for (4) which for the general case is difficult to find (Vujanovic and Jones, 1989) but it can be easier if we consider cases where the generalized linear momentum $p = \partial L/\partial v$ has the following forms

p = **p**(**x**, **v**): If the generalized linear momentum is independent of time, then Eq. (4) turns into

$$v\frac{\partial G}{\partial x} + F\frac{\partial G}{\partial v} + \frac{\partial F}{\partial v}G = 0 \tag{5}$$

where $G = \partial^2 L/\partial v^2$. The general solution for (5) is given by

$$\frac{\partial^2 L}{\partial v^2} = \exp\left(\int \frac{\partial F}{\partial v}\, dt\right), \tag{6}$$

and the Lagrangian is obtained through the integration

$$L(x, v) = \int dv \int G(x, v)\, dv + f_1(x)v - f_2(x), \tag{7}$$

where $f_1(x)$ and $f_2(x)$ are arbitrary functions. The second term on the right side of (7) corresponds to a gauge of the Lagrangian which brings about an equivalent Lagrangian (Goldstein, 1980), and it is possible to forget it. Using (7) in (2) the Hamiltonian of the systems is obtained and it is a constant of the motion since it is time independent.

p = **p**(**t**, **v**): If the generalized linear momentum is independent of position, then Eq. (4) turns into

$$F\frac{\partial^3 L}{\partial v^3} + \frac{\partial^3 L}{\partial v^2 \partial t} + \frac{\partial F}{\partial v}\frac{\partial^2 L}{\partial v^2} = 0, \tag{8}$$

which means that

$$\frac{\partial}{\partial v}\left(F\frac{\partial^2 L}{\partial v^2} + \frac{\partial^2 L}{\partial t \partial v}\right) = 0,$$

the term in parenthesis is dp/dt, therefore if the generalized linear momentum is independent of position then the Euler-Lagrange equation must be of the form

$$\frac{dp}{dt} = f(t, x), \tag{9}$$

where $f(t, x)$ is the generalized force and may be any arbitrary function of position and time. Writing the generalized linear momentum as $p = \mu(t)g(v)$ then Eq. (9) is given by

$$\frac{d\mu}{dt}g(v) + \mu(t)\frac{dg}{dv}\frac{dv}{dt} = f(t, x), \tag{10}$$

if we are dealing with a relativistic system with linear dissipation, then

$$\frac{dv}{dt} = \left(-\frac{\partial U(t, x)}{\partial x} - \alpha v\right)(1 - v^2/c^2)^{3/2}, \tag{11}$$

where α is a positive real parameter, c is the speed of light and $U(t, x)$ represents the potential energy of the system. Substituting (11) into (10) we have

$$\frac{d\mu}{dt} g(v) + \mu(t) \frac{dg}{dv} \left(-\frac{\partial U(t, x)}{\partial x} - \alpha v \right) (1 - v^2/c^2)^{3/2} = f(t, x), \tag{12}$$

if

$$\frac{dg}{dv} = \frac{1}{(1 - v^2/c^2)^{3/2}}$$

then Eq. (12) is given by

$$v \left(\frac{1}{\sqrt{1 - v^2/c^2}} \frac{d\mu}{dt} - \alpha \mu(t) \right) - \mu(t) \frac{\partial U}{\partial x} = f(t, x), \tag{13}$$

since Eq. (13) must be velocity independent then

$$\frac{1}{\sqrt{1 - v^2/c^2}} \frac{d\mu}{dt} = \alpha \mu(t), \tag{14}$$

the solution for (14) is given by

$$\mu(t) = \exp\left(\alpha \int \sqrt{1 - v^2/c^2}\, dt \right) = e^{\alpha \tau(t)}, \tag{15}$$

where

$$\tau(t) = \int \sqrt{1 - v^2/c^2}\, dt, \tag{16}$$

is known as the proper time of the particle (Goldstein, 1980). Therefore Eq. (11) can be expressed as

$$\frac{d}{dt} \left(\frac{v e^{\alpha \tau}}{\sqrt{1 - v^2/c^2}} \right) = -e^{\alpha \tau} \frac{\partial U}{\partial x}, \tag{17}$$

and the Lagrangian and Hamiltonian for (17) are given by

$$L(t, x, v) = -c^2 e^{\alpha \tau} \sqrt{1 - v^2/c^2} - e^{\alpha \tau} U(t, x), \tag{18}$$

$$H(t, x, p) = c^2 e^{\alpha \tau} \sqrt{1 + p^2 e^{-2\alpha \tau}/c^2} + e^{\alpha \tau} U(t, x). \tag{19}$$

If $v \ll c$ then (19) reduces to the so called Caldirola-Kanai Hamiltonian

$$H(t, x, p) \approx \frac{p^2 e^{-\alpha t}}{2} + e^{\alpha t} U(t, x) + c^2 e^{\alpha t}, \tag{20}$$

which describes the motion of a nonrelativistic particle with linear dissipation (Um *et al.*, 2002).

3. APPROXIMATE RELATIVISTIC HAMILTONIAN

Consider the equation for a free relativistic particle subject to a dissipative force which is proportional to its velocity, the equation of motion of this system is given by

$$\frac{dv}{dt} = -\alpha v(1 - v^2)^{3/2} \tag{21}$$

where α is a positive constant and we have set $c = 1$. The Hamiltonian for (21) is given by

$$H(t, x, p) = e^{\alpha\tau}\sqrt{1 + p^2 e^{-2\alpha\tau}} \tag{22}$$

So far, everything is exact, but we have to specify $\tau(t) = \int \sqrt{1 - v^2}\, dt$, therefore we have to integrate the equation of motion (21), doing this we get

$$C_0 - \alpha t = \frac{1 - \xi \tanh^{-1} \xi}{\xi}, \tag{23}$$

where $\xi = \sqrt{1 - v^2}$ and C_0 is the integration constant which can be easily determined imposing that $v(0) = v_0 \neq 0$. Expanding the term $\tanh^{-1} \xi$ and taking into account only terms less than or equal to ξ^3, then Eq. (23) turns into

$$\xi^2 + \beta(t)\xi - 1 = 0, \tag{24}$$

where $\beta(t) = C_0 - \alpha t$, solving Eq. (24) we obtain

$$\xi(t) = \frac{-\beta(t) \pm \sqrt{\beta(t)^2 + 4}}{2}, \tag{25}$$

we have two solutions for $\xi(t)$, but we can get rid of one knowing that $\xi(t)$ is a positive defined function, therefore we must choose

$$\xi(t) = \frac{-\beta(t) + \sqrt{\beta(t)^2 + 4}}{2}. \tag{26}$$

Integrating (26) with respect to time we have

$$\tau(t) = \frac{\beta(t)^2}{4\alpha} - \frac{\beta(t)}{4\alpha}\sqrt{\beta(t)^2 + 4} - \frac{1}{\alpha}\sinh^{-1}(\beta(t)/2), \tag{27}$$

substituting (27) into (22) we have the approximate relativistic Hamiltonian for the equation of motion (21).

4. CONCLUSIONS

A general formalism to obtain the Lagrangian and Hamiltonian for one dimensional dissipative systems was obtained. The Lagrangian and Hamiltonian for a relativistic particle with linear dissipation was deduced. An example of this approach was given for the case of a relativistic particle with linear damping.

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